

Transition from Poissonian to Gaussian-orthogonal-ensemble level statistics in a modified Artin's billiard

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One wall of an Artin's billiard on the Poincaré half-plane is replaced by a one-parameter (c_p) family of nongeodetic walls. A brief description of the classical phase space of this system is given. In the quantum domain, the continuous and gradual transition from the Poisson-like to Gaussian-orthogonal-ensemble (GOE) level statistics due to the small perturbations breaking the symmetry responsible for the "arithmetic chaos" at $c_p = 1$ is studied. Another GOE \rightarrow Poisson transition due to the mixed phase space for large perturbations is also investigated. A satisfactory description of the intermediate level statistics by the Brody distribution was found in both cases. The study supports the existence of a scaling region around $c_p = 1$. A finite-size scaling relation for the Brody parameter as a function of $1 - c_p$ and the number of levels considered can be established.

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I. INTRODUCTION

One of the fundamental questions of quantum chaos is the relationship between the different types of classical motion and the energy-level statistics of the quantum counterpart of the system [1-3]. One important field for testing ideas on the subject is the geodesic flow on constant-negative-curvature surfaces [4-6]. The most often studied cases are the symmetric octagon and the modular surface [4,5,7-9]. The level-spacing statistics of these systems were not found to follow the predictions of the random matrix theory for the Gaussian orthogonal ensemble (GOE) [4,5,7,8]. This is contrary to the general belief that completely chaotic systems with time-reversal symmetry have GOE level-spacing statistics [10,3,11]. This unexpected feature was explained recently [12,9] based on semiclassical arguments. Unlike generic ergodic systems, there is an exponential degeneracy of time periods [13] of closed classical orbits caused by nontrivial symmetries [12], which is responsible for the amplification of the quantum interference in these systems [9].

In this paper we investigate a modification of Artin's

billiard introduced in Ref. [8], where one geodetic wall is replaced with a one-parameter family of nongeodetic walls. The advantage is that changing only one parameter we can study perturbations of this billiard. The perturbation breaks the degeneracy of time periods and the transition from Poissonian to GOE level spacing distribution can be followed. At parameter values corresponding to large perturbations another transition from GOE to Poissonian statistics can be observed due to the change of the ergodic part of the mixed phase space up to complete integrability [14].

The paper is organized as follows. In Sec. II we introduce the family of the billiards, and give a brief description of the classical dynamics. In Sec. III we shall investigate the manifestation of the classical dynamics in the statistical properties of the energy levels. Finally, Sec. IV is devoted to conclusions.

II. CLASSICAL TREATMENT

Our billiard is given in the Poincaré half plane with Gaussian curvature $K = -1$ and Riemannian scalar curvature $R = -2$. The motion between two bounces can be obtained from the classical Lagrangian

$$L = \frac{1}{2y^2}(\dot{x}^2 + \dot{y}^2) \quad (1)$$

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or Hamiltonian

$$H = \frac{1}{2}y^2(p_x^2 + p_y^2), \quad (2)$$

where

$$p_x = \frac{\dot{x}}{y^2}, \quad p_y = \frac{\dot{y}}{y^2}. \quad (3)$$

The domain of the classical motion is the area bounded by the vertical walls

$$x = 0, \quad x = 1/2 \quad (4)$$

and the circle

$$x^2 + (y - 1 + R)^2 = R^2 \quad (5)$$

with radius R , and center $(0, 1 - R)$. We use the inverse radius $c_p = 1/R$ as a control-parameter. The collisions with the walls are elastic. The trajectories of the free motion between the bounces are circles centered on the x axis. The footpoint (x) and the inverse radius ($M = 1/r$) of the trajectories (Fig. 1) is a good choice for the “bounce mapping” T . The pair (x, M) remains constant during the free motion and it changes into a new pair (x', M') at each bounce,

$$(x, M) \rightarrow (x', M') = T(x, M). \quad (6)$$

We have studied the classical dynamics in the parameter range $0 \leq c_p \leq 2$. Note that for $c_p > 2$ the bottom of the billiard is open. In the classical treatment we measure the time in $(2E)^{-1/2}$ units. Then the elapsed time agrees with the metric length of orbits. The billiard at $c_p = 1$ is the domain of Artin’s triangle billiard and is bounded by three geodesic walls. At $c_p = 0$ the billiard has a horizontal nongeodesic straight segment, and the motion is integrable [15,14].

In the parameter range $1 \leq c_p < 2$ the motion is completely chaotic; the Lyapunov exponent is positive for

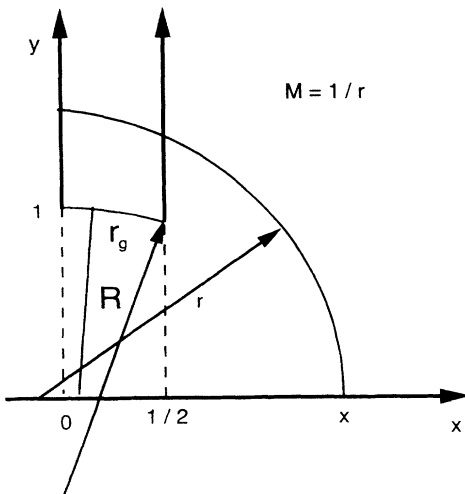


FIG. 1. Our billiard model on the Poincaré half plane.

each periodic orbit as we will show next. For $c_p = 1$ this is a consequence of Artin’s proof for the ergodicity [16]. For other parameters we have used a method common in the mathematical literature [17–20] for the calculation of the Lyapunov exponent of the trajectories. Namely, following the time evolution of the local curvature $\kappa(t)$ of an orthogonal curve along a trajectory and choosing $\kappa(0) > 0$ the quantity $\exp[\int_0^T \kappa(t)dt]$ measures the stretching orthogonal to the orbit [17]. Consequently, the Lyapunov exponent of the orbit is given by the average value of κ ,

$$\lambda = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \kappa(t)dt. \quad (7)$$

During the free motion $\kappa(t)$ is described by the Riccati equation [18]

$$\dot{\kappa}(t) = -K(t) - \kappa^2(t), \quad (8)$$

where $K(t)$ is the local Gaussian curvature, which has a constant value of $K(t) = K = -1$ in our case. The change of κ before and after the collision with a wall having curvature q in the place of the bounce is given by

$$\kappa_+ = \kappa_- - \frac{2q}{\cos \varphi}, \quad (9)$$

where φ is the angle of the bounce. If the curvature q is negative or zero for each wall, it is easy to see from the solution of (8) that after sufficient but finite time (T_{\max}) κ becomes larger than 1 [21], and then $\kappa(t) \geq +1$ holds for $t \geq T_{\max}$. The average of κ also becomes larger than +1 which means that the system is hyperbolic. This is the case when $c_p > 1$, since the curvature of the bottom wall is

$$q = \frac{1}{r_g} - \frac{1}{R}, \quad (10)$$

where r_g is the radius of the geodesic touching the wall in the point where the curvature is computed (Fig. 1). The quantity (10) is negative or zero, if $1 \leq c_p = 1/R$. The Lyapunov exponent is $\lambda = 1$ if $c_p = 1$, and $\lambda \geq 1$ if $c_p > 1$. Since these arguments hold for each orbit, the system cannot contain stable periodic orbits.

In the remaining parameter range $0 < c_p < 1$ it is not obvious for which parameter values the system is completely chaotic, and where the first island of stability is created. The bottom wall is locally focusing in this parameter range [22]. We expect that the highly unstable free motion suppresses the focusing on a global scale up to a certain strength of the local focusing. In other words, there exists a parameter $c_p^* < 1$ so that for $c_p^* < c_p < 2$ the system is completely chaotic.

To have an insight into how the mixed phase space sets in we have investigated analytically the stability of a family of periodic orbits starting at the left corner of the billiard and bouncing m times on the vertical walls (Fig. 2). This family of orbits exists in the whole $0 \leq c_p \leq 2$ parameter range. They are obviously unstable for $c_p > 1$. We shall show later that for $c_p < 1$ their stability criteria

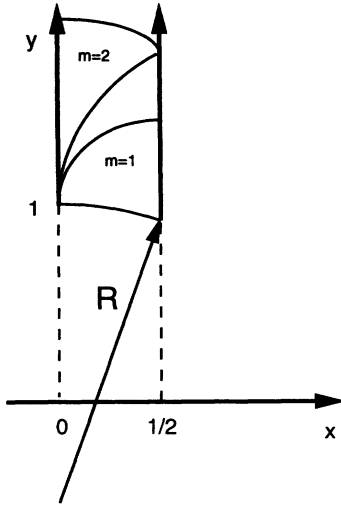


FIG. 2. Periodic orbits bouncing once in the left corner and m times on the vertical walls.

give a good insight into the structure of the mixed phase space. Varying the parameter, the shape of these orbits remains unchanged. The solution of the Riccati equation (8) for $c_p < 1$ and $0 < \kappa(0) < 1$ between two collisions with the bottom wall reads as

$$\kappa_m(t) = \tanh(t - t_0^{(m)}), \quad (11)$$

where $t_0^{(m)}$ is a parameter to be adjusted. According to (9) bouncing with the bottom wall, the curvature changes as

$$\kappa_m(t+0) = \kappa_m(t-0) - Q. \quad (12)$$

For the family of the orbits considered here Q takes the form

$$Q = \frac{2(1 - c_p)}{\cos \varphi_m}, \quad (13)$$

where $1 - c_p$ is the curvature of the bottom wall in the bounce point, and φ_m is the collision angle of the orbit labeled by m . A direct calculation leads to the equation

$$\cos \varphi_m = \frac{m/2}{\sqrt{1 + m^2/4}}. \quad (14)$$

Since according to (7) the Lyapunov exponent requires the average of $\kappa(t)$ only in the limit when T goes to infinity, one can calculate this with the asymptotic solution of $\kappa(t)$ which satisfies the periodicity condition

$$\kappa_m(t + T_m) = \kappa_m(t), \quad (15)$$

where T_m is the period of the orbit given by

$$\cosh(T_m/2) = \sqrt{1 + m^2/4} \quad (16)$$

from simple geometric considerations. Solving (15) with the condition (12) leads to the equation

$$\tanh(-t_0^{(m)}) = \tanh(T_m - t_0^{(m)}) - Q \quad (17)$$

for $t_0^{(m)}$. This is a second order equation for $\tanh(t_0^{(m)})$. Its relevant solution is

$$\tanh(t_0^{(m)}) = \frac{1}{m} \left[-\sqrt{c_p^2(m^2 + 4) - 4c_p} + (1 - c_p)\sqrt{4 + m^2} \right]. \quad (18)$$

The Lyapunov exponent is the average of κ for one period:

$$\begin{aligned} \lambda_m &= \frac{1}{T_m} \int_0^{T_m} \tanh(t - t_0^{(m)}) dt \\ &= \frac{1}{T_m} \ln[\cosh(T_m - t_0^{(m)}) / \cosh(t_0^{(m)})]. \end{aligned} \quad (19)$$

From Eqs. (19) and (18) one can calculate the Lyapunov exponent as a function of c_p and the index of the orbit m analytically. Figure 3 shows the Lyapunov exponent

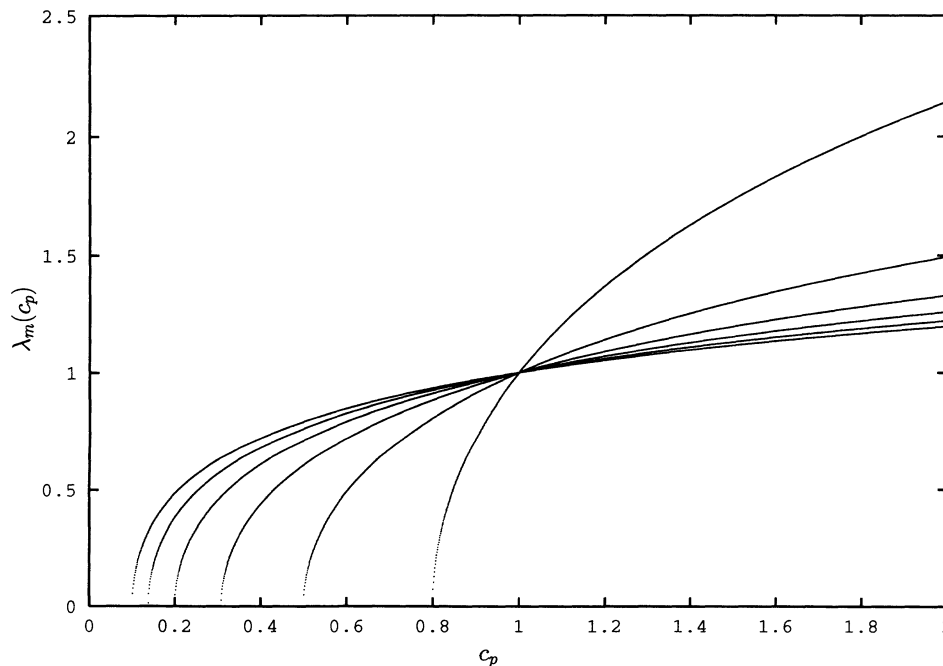


FIG. 3. Lyapunov exponents as a function of the c_p parameter, for the periodic orbits of Fig. 2. Only nonzero Lyapunov exponents are plotted.

as a function of c_p for this family. For $c_p = 1$ all the orbits have the same Lyapunov exponent ($\lambda_m = +1$) as a well known consequence of the geodetic walls. Using Eqs. (19) and (18) one can also solve the equation

$$\lambda_m(c_p) = 0 \quad (20)$$

and get that parameter value c_p where the m th orbit of the family gets stabilized when c_p is decreased,

$$c_p = \frac{4}{m^2 + 4}. \quad (21)$$

First the shortest orbit ($m = 1$) becomes stable at $c_p = 0.8$.

To decide whether there exists another periodic orbit, which is stable for a larger parameter value c_p , we have investigated the bounce mapping (6) (see Fig. 4). We have not found any island in the range $0.8 < c_p < 1$. The existence of a stable orbit causing a very small island, of course, cannot be excluded, but the phase space in this range can be considered completely chaotic at least practically.

After this description of the classical domain, we turn to the quantum treatment of the system.

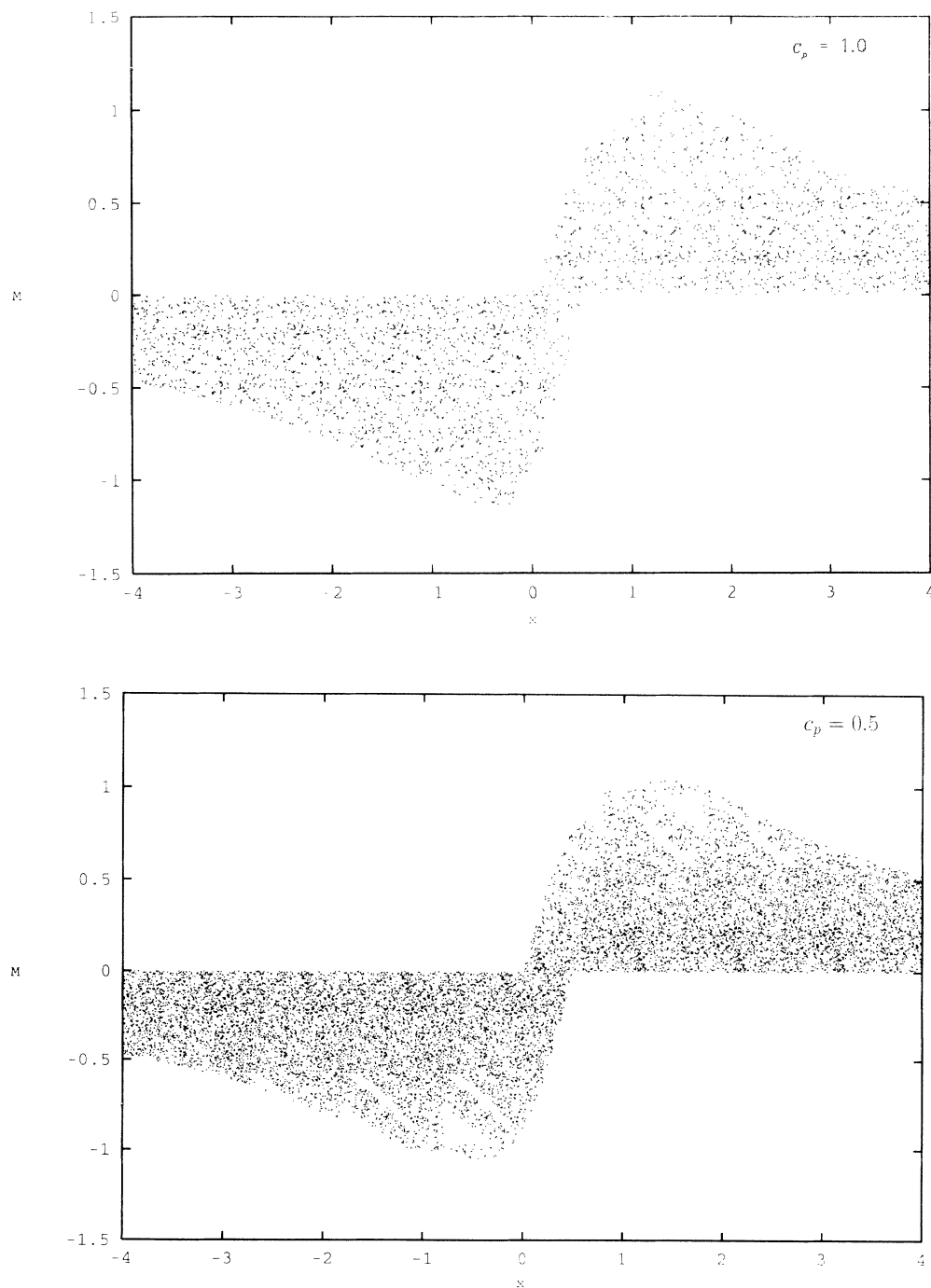


FIG. 4. The phase space of the bounce mapping (6) for $c_p = 1, 0.5$, and 0.07 are shown. Notice that there is no island of stability for a certain $M(c_p) > |M|$ strip.

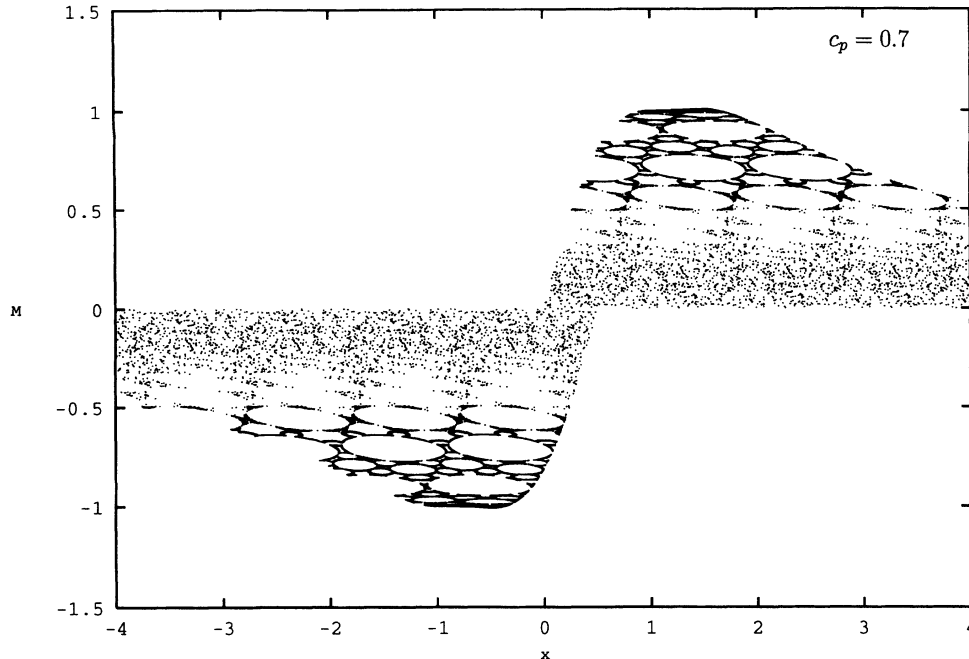


FIG. 4 (Continued).

III. LEVEL STATISTICS AND SCALING PROPERTIES

The energy eigenvalues of the billiard can be computed by solving the Schrödinger equation

$$-\frac{1}{2}y^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = E\psi \quad (22)$$

with the Dirichlet ($\psi = 0$) boundary condition on the walls (4) and (5). In the integrable case ($c_p = 0$) the partial differential equation (22) is separable in x and y , and the eigenfunctions can be derived [14]. Using this functional basis for the nonintegrable problem, the Schrödinger equation (22) can be solved numerically. For further details we refer to our earlier papers on this model [15,14,8]. The unfolding of the levels has been performed by the transformation

$$x_i = \bar{N}(E_i), \quad (23)$$

where $\bar{N}(E)$ is given by the Weyl formula of Ref. [8].

Once one has the unfolded eigenenergies, one can analyze their fluctuations. We concentrated on the distribution $p(s)$ of spacings between nearest neighbor unfolded levels ($s = x_{i+1} - x_i$), since this has been found to be very sensitive to the classical dynamics [23,1,11]. The distribution $p(s)$ and its first momentum are normalized due to the unfolding (23):

$$\int_0^\infty p(s) ds = 1, \quad \int_0^\infty s p(s) ds = 1. \quad (24)$$

It is expected, that the low lying energy levels are determined by the quantum nature of the system while the higher semiclassical part of the spectrum reflects the classical properties, and exhibits universal behavior for

integrable and completely chaotic systems. Generic integrable systems show Poissonian distribution [24–27],

$$p(s) = e^{-s}. \quad (25)$$

In the completely chaotic case the spectra can be described by ensembles of random matrices, namely the spacing distribution of the GOE applies [28]. This distribution is almost identical with the Wigner surmise,

$$p(s) = \frac{\pi s}{2} e^{-\pi s^2/4}. \quad (26)$$

In the case of mixed phase space the situation is less clear, the fluctuations are nonuniversal, and the discussion about relevant parameters characterizing the level-spacing distribution of these systems is not finished yet.

It is of particular interest that at $c_p = 1$ the level statistics do not follow the generic behavior of classically chaotic systems. In this “arithmetic” case [9,12] there exist arguments for the relevance of the Poissonian distribution based on the Selberg trace formula and on the asymptotic distribution of time periods of periodic orbits [9]. However, these considerations are valid for the asymptotic, infinitely high part of the spectrum. In the statistics of the first, say, 2000 levels deviations from the Poissonian statistics can be observed [8,9]. For parameters slightly different from $c_p = 1$ the universal GOE distribution should show up immediately in the asymptotic, infinitely high part of the spectrum, if the universality of GOE for completely chaotic systems is true. The change from Poissonian to GOE will gradually extend towards the lower lying levels, when further decreasing c_p .

According to the classical behavior of the system and the considerations above, we can expect three regions in the parameter space where the nearest neighbor level spacing distribution of a finite number of levels behaves

differently. The first region is $0 \leq c_p < 0.8$, where, with the exception of the integrable case $c_p = 0$ in which the phase space is not mixed, one can expect some intermediate statistics between GOE and Poissonian. The second region is $0.8 < c_p < 2$, $c_p \neq 1 \pm \Delta$, where GOE is expected. The narrow region around $c_p = 1$, indicated by Δ , is a third region, where we should observe certain deviations from the GOE for any finite number of levels due to the “arithmetic chaos” in $c_p = 1$.

The first and third regions require a family of distributions interpolating between the Poissonian and the GOE for quite different reasons. There are several attempts to find a theoretically well established transition formula for systems with mixed phase space [29,30]. One of the main differences in these crossover formulas is in the small s behavior of $p(s)$. The Berry-Robnik formula [29] gives a constant value for small s [$p(s) \sim \text{const}$], while the Lenz-Haake formula [30] leads to a linear level repulsion s [$p(s) \sim s$].

There are other formulas which interpolate between (26) and (25) on a purely empirical base. One of them was proposed by Israilev [31] and another one by Brody [32]. Since they are not necessarily associated with a mixed phase space, they can be tried also in the third region. Both of them show power-law level repulsion for small s [$p(s) \sim s^\beta$]. To characterize intermediate distributions we have chosen the one-parameter family of Brody distributions since it is simpler than the other one and, for other purposes, we could not see any advantage of Israilev’s distribution. Moreover, in a recent study [33] both of them were found equally satisfactory. The one-parameter family of Brody distributions is given by

$$p(s, \beta) = as^\beta \exp(-bs^{\beta+1}), \quad (27)$$

where a and b can be derived from the conditions (24) as a function of the Brody parameter β ,

$$a = (\beta + 1)b, \quad b = \left[\Gamma \left(\frac{\beta + 2}{\beta + 1} \right) \right]^{\beta+1}. \quad (28)$$

For $\beta = 0$ Eq. (25) and for $\beta = 1$ Eq. (26) are recovered. The cumulative level density of the Brody distribution is

$$I(s) = \int_0^s p(x) dx = 1 - \exp(-bs^{\beta+1}). \quad (29)$$

We have fitted Brody distributions to our empirical $I(s)$ functions applying the method of least squares. Using the first n energy levels at a certain c_p and finding the best fit yields an empirical $\beta_n(c_p)$ value. Note that we do not claim that the Brody distribution is the correct asymptotic form, but only that the fitted β parameter is a good measure where the intermediate distribution is lying between (25) and (26). We have carried out this program in the parameter region $0 \leq c_p \leq 1.2$, calculating about 2000 energy levels. We have found that the Brody distribution provides a satisfactory fit in the whole parameter region

In Fig. 5., where the Brody parameter β as a function of c_p is depicted, one can see that β increases continuously with c_p in the region $0 \leq c_p < 0.8$, reflecting the

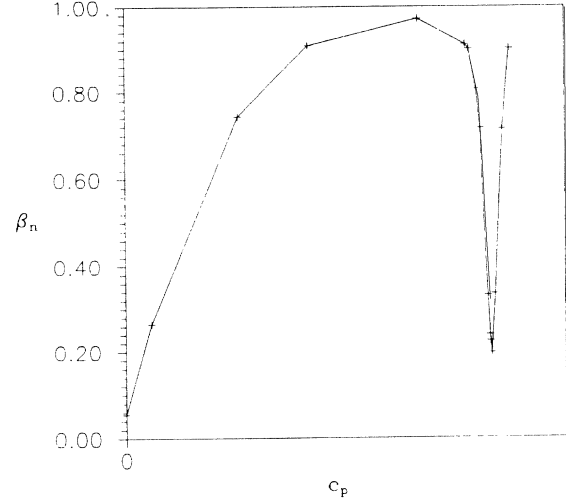


FIG. 5. The Brody parameter as a function of the control parameter c_p . Each point, denoted by a + sign, was calculated from the first 1947 quantum levels.

fact that in the phase space the volume of the chaotic region increases. This is in complete agreement with earlier findings of Ref. [34] in another system with mixed phase space. The saturation of β starting at $c_p \approx 0.5$ indicates that the level-spacing distribution is not sensitive to small stable islands. A relatively sharp transition from GOE toward Poissonian can be observed in the $0.1 > |1 - c_p|$ parameter range, which we have studied in more detail.

In the critical point $c_p = 1$ the numerical level spacing distribution is still far from the theoretical Poissonian distribution. To see the convergence of the Brody parameter towards $\beta(c_p = 1) = 0$ we have plotted $\beta_n(c_p = 1)$ as a function of the first n energy levels (Fig. 6). The convergence is quite apparent and β_n can be well approximated in the range $n = 200 - 2000$ as

$$\beta_n(c_p = 1) = 13.05n^{-\alpha} \quad (30)$$

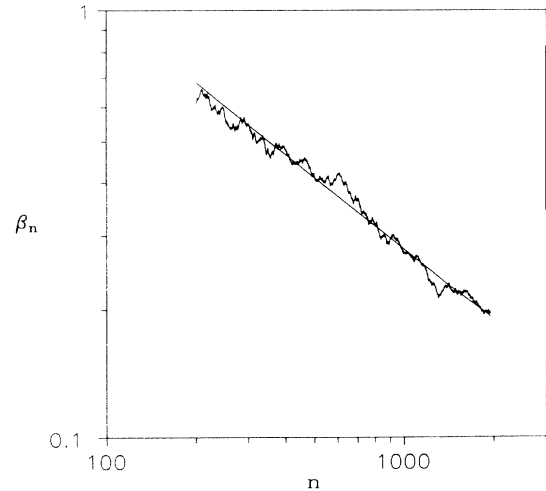


FIG. 6. The Brody parameter as a function of the number of levels on a log-log plot at parameter $c_p = 1$ corresponding to the arithmetic case. The dashed line represents the best power-law fit relation $\beta_n = 13.05n^{-0.55}$.

with $\alpha \approx 0.55$. This is very close to an $\sim 1/\sqrt{n}$ dependence, which is typical in random matrix description of transitions between ensembles [35]. When $c_p \equiv 1 - \delta < 1$ we observed for sufficiently small δ values ($\delta < 0.01$) the following behavior of β as function of n . For small n numbers [$n < n^*(\delta)$] the $\beta_n(c_p = 1 - \delta)$ curves coincide and they tend to constant values for $n > n^*(\delta)$ within the $n < 2000$ range (Fig. 7). These values are scaling with δ as

$$\beta_n(1 - \delta) \sim \delta^\gamma, \quad (31)$$

where $\gamma \approx 0.44$. The separate scaling relations (30) and (31) can be unified into one finite size scaling relation familiar from the theory of phase transitions,

$$\frac{1}{\beta_n(1 - \delta)} = \delta^{-\gamma} f(1/\delta n^\epsilon), \quad (32)$$

where this relation defines the scaling function $f(x)$. For $x \rightarrow 0$ the scaling function goes to a finite value, while for $x \rightarrow \infty$ it behaves like $f(x) \sim x^{-\gamma}$. The n dependence (30) can be recovered by choosing $\epsilon = \alpha/\gamma$. The existence of such a formula is stronger evidence of the scaling than the two scaling relations (30) and (31) together, since the scaling functions computed from different data must coincide. This has been checked at three different δ parameters. Satisfactory agreement with the scaling hypothesis (32) has been found within the statistical fluctuations of the scaling functions (see Fig. 8).

Since our billiard system is expected to be completely chaotic for small δ values, a finite $\beta \neq 1$ asymptotic value would mean that the GOE statistics is not universal. We think instead that the finite value of β is restricted to a region $n^*(\delta) \ll n \ll n^{**}(\delta)$ only. The emergence of such a region can be understood in terms of semiclassical considerations. The strength of the perturbation of the arithmetic billiard is proportional to $\delta = 1 - c_p$. Let us denote by $S_p(E)$ the action belonging to the periodic

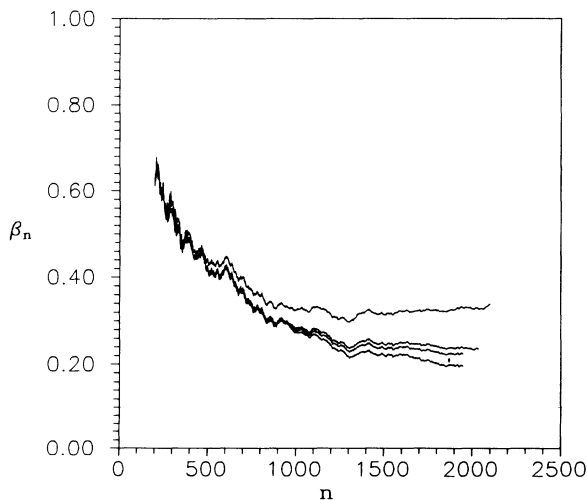


FIG. 7. The Brody parameter as a function of the number of levels. The curves are taken at $c_p = 1.000, 0.996, 0.995$, and 0.990 . The bigger the c_p the lower lying the corresponding curve.

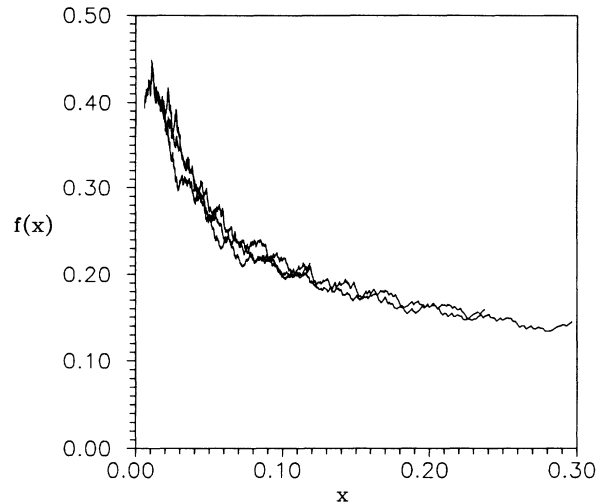


FIG. 8. The scaling function $f(x)$ reconstructed from $\beta_n(c_p)$ at parameters $c_p = 0.990, 0.995$, and 0.996 . Notice that the curves coincide within their fluctuations.

orbit p . The perturbation destroys the degeneracy of the actions $S_p(E)$. If two actions in the arithmetic billiard are degenerate $S_p(E) = S_{p'}(E)$, $p \neq p'$, they differ in an amount of

$$\Delta S_{p,p'}(\delta, E) = S_p(\delta, E) - S_{p'}(\delta, E) = (2E)^{1/2} l_{p,p'}(\delta) \quad (33)$$

in the perturbed case, where $l_{p,p'}(\delta) = l_p(\delta) - l_{p'}(\delta)$ is the difference of the metric length of the periodic orbits p and p' . $l_{p,p'}(\delta)$ is independent of the energy. The absence of the degeneracy of the periodic orbits p and p' can be observed in physical quantities when the difference of actions is comparable to \hbar (in our units $\hbar = 1$):

$$\Delta S_{p,p'}(\delta, E) \approx \hbar. \quad (34)$$

Because of the energy dependence (33) of this relation, at low energies (small n) the absence of the degeneracy of actions cannot be observed. The condition (34) gives a critical energy value to each pair of periodic orbits, and their minimum defines $n^* \sim E_{\min}/d$, where d is the mean level spacing [8]. This argument also explains that all the $\beta_n(1 - \delta)$ curves coincide for $n \ll n^*$. For $\delta = 0$ this region extends to the infinite n range. When $\delta > 0$ for sufficiently high energy (large n) the broken degeneracy of all periodic orbits can be observed and the system looks like an ordinary completely chaotic system. This defines another scale n^{**} depending on c_p . Between n^* and n^{**} we expect to observe roughly a constant value of β_n . From n^{**} , the β_n function is supposed to converge toward $\beta = 1$. For $c_p \rightarrow 1$, n^{**} will grow to infinity, and the length of the scaling region is increasing too. However, for c_p sufficiently far from 1 the scaling region is very restricted or disappears ($n^* \approx n^{**}$). At parameter $\delta = 0.03$ instead of a plateau, β_n has merely a minimum around $n \approx 700$ (Fig. 9).

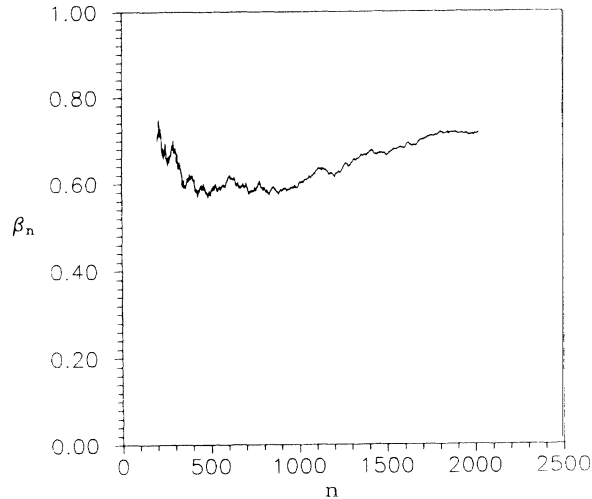


FIG. 9. The Brody parameter as a function of the number of levels at $c_p = 0.970$. Notice the very narrow scaling region around $n = 700$.

IV. CONCLUSIONS

In this paper we have analyzed the connection of classical dynamics and the statistical properties of the quantum energy spectrum in a constant negative curvature billiard family with a nongeodetic wall. In cases where the phase space is mixed, we have found remarkable

agreement with the Brody distribution which interpolates between GOE and Poissonian. The Brody parameter was found to be a monotonically increasing function of the chaoticity of the system as measured by the magnitude of the positive Lyapunov exponent. Using the Brody parameter we could describe a GOE \rightarrow Poissonian transition around $c_p = 1$ due to the arithmetic chaos. We observed a finite size scaling relation among the Brody parameter, the number of energy levels, and the parameter $1 - c_p$, and argued that the existence of the scaling region can be explained by the energy dependence of the condition (34). We hope the detailed quantitative theory of this scaling behavior can be worked out in the future.

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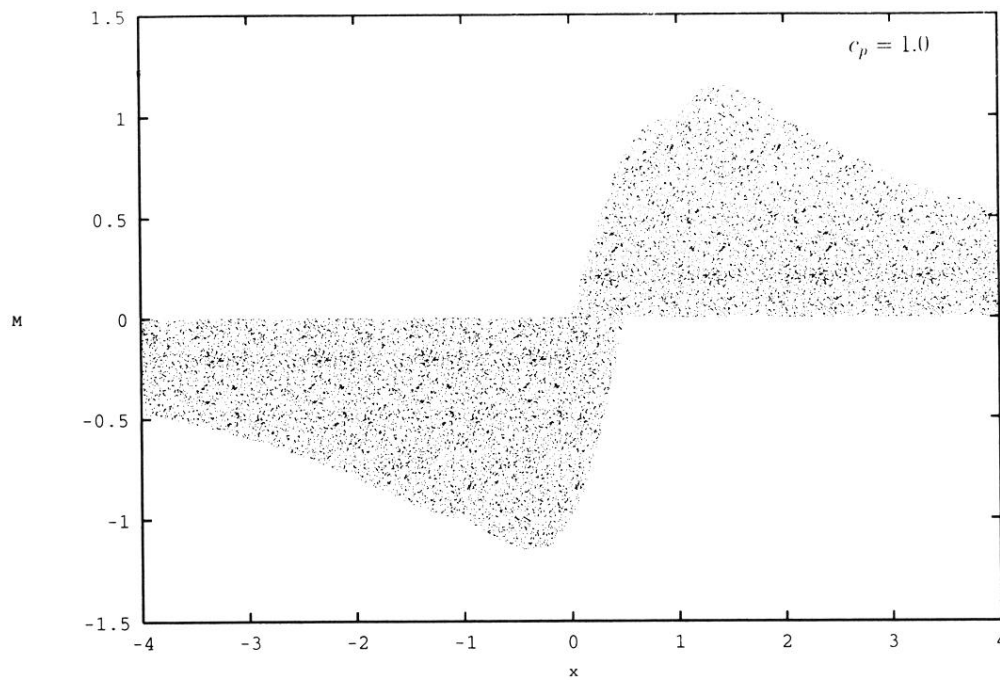


FIG. 4. The phase space of the bounce mapping (6) for $c_p = 1, 0.5$, and 0.07 are shown. Notice that there is no island of stability for a certain $M(c_p) > |M|$ strip.

